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A Basis for the Local Solutions of an Elliptic Equation

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With a view toward applications to eigenfunction expansions and spectral asymptotics for partial differential operators with continuous spectra, the authors study the problem of characterizing a solution of a given second-order elliptic linear differential equation by its behavior at a given point. When the elliptic operator is the Laplacian plus lower-order terms, and the coefficients of those terms are sufficiently smooth in the angular directions about the chosen point, the classification of harmonic functions by their local behavior (via spherical harmonics) can be carried over intact to the solutions of the more general equation, because local solutions of the two equations can be placed in one-to-one correspondence. Under this correspondence, the images of the standard basis of harmonic polynomials constitute a basis for expanding every solution that is definable in some neighborhood (no matter how small) of the point.

1. INTRODUCTION

Although unique continuation theorems (see [13, Section 19] for references) show that the solutions of a second-order elliptic partial differential equation can be uniquely characterized by data at a single point, they do not indicate what constitutes a complete, nonredundant set of data. That question could be answered by exhibiting a basis for the space of local solutions (germs) of the equation; that is, a set of functions such that (a) every solution definable in any neighborhood (no matter how small) of the given point may be expanded in a series of these functions; (b) the expansion coefficients are unique and depend only on the behavior of the solution near the point. Given the local characterization of solutions by these coefficients, one could develop the theory of eigenfunction expansions for self-adjoint partial differential operators with (possibly) continuous spectrum along the lines of the Weyl–Titchmarsh–Kodaira theory [16, 17, 10] for ordinary differential operators.

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For harmonic functions, such a basis is well known: A complete, linearly independent set, $\{H_{l,m}(\mathbf{x})\}$, of harmonic polynomials ($H_{l,m}$ being homogeneous of degree l) is a basis for the space of solutions of $\Delta\phi = 0$ in neighborhoods of the origin. One approach to constructing a basis in the case of a more general second-order linear elliptic equation is to set up a one-to-one correspondence between functions harmonic near $\mathbf{0}$ and the local solutions of the other equation near the distinguished point; each basis element is then the image of one of the $H_{l,m}$.

Gilbert [7, Theorem 6.2] takes this approach to obtain a basis for the local solutions of the following self-adjoint, strongly elliptic equation with Hölder-continuous coefficients:

$$\sum_{j,k=1}^{3} \frac{\partial}{\partial x_j}\left(a_{jk}(\mathbf{x})\frac{\partial \phi}{\partial x_k}\right) = 0.$$

The mapping he constructs is an integral operator analogous to the Whittaker–Bergman operator [6, Chap. 2]. Bergman [1], Colton [2], and others (see bibliography in [7]) have used similar integral operators to set up such correspondences for various classes of equations in two to four variables. Unfortunately, for equations with five or more variables, Kukral [11] has shown that such integral operators do not exist.

In the present paper a correspondence of local solutions with harmonic functions, leading to a basis of the desired type, is constructed by a more direct method depending heavily on the expansion of functions in spherical harmonics. The type of equation treated is

$$\left[-\Delta + \sum_{j>k=1}^{n} \beta_{jk}(\mathbf{x})\left(x_k\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_k}\right) + V(\mathbf{x})\right]\phi = 0, \tag{1.1}$$

where Δ is the ordinary Laplacian and the dimension, n, is arbitrary. The coefficients of the lower-order terms satisfy conditions of angular smoothness discussed in Section 3, but may be badly behaved in the radial direction.

The first-order term in (1.1) has been written in a form convenient for analysis in terms of spherical harmonics about the distinguished point, which has been taken to be $\mathbf{0}$. The general first-order operator

$$\sum_{j=1}^{n} b_j(\mathbf{x})\frac{\partial}{\partial x_j}$$

is reduced to that form in Section 2 by a change of dependent variable.

Two families of function spaces are introduced in Section 3. One of these serves to provide the underlying spaces in which (1.1) will be solved; the other comprises the spaces from which the coefficients in (1.1) will be drawn. Two inverses of the Laplacian, with properties of the sort associated with Green functions and influence functions (retarded Green functions), respectively, are constructed and investigated at the end of that section.

In the first half of Section 4, a correspondence between certain harmonic functions and certain solutions to (1.1) is found; it is then used to construct the basis. The second half is devoted to proving that this basis can be used to expand all functions which, in a sufficiently small ball about **0**, both solve (1.1) and have square-integrable second derivatives.

2. Reduction of the First-Order Terms

The most general equation with the Laplacian as the leading term is of the form

$$-\Delta\phi + \mathbf{b}\cdot\nabla\phi + V\phi = 0, \qquad \mathbf{b}\cdot\nabla\phi \equiv \sum_{j=1}^{n} b_j \frac{\partial\phi}{\partial x_j}, \tag{2.1}$$

where V and b_j are functions of $\mathbf{x}$. If

$$\mathbf{x}\cdot\mathbf{b}(\mathbf{x}) = 0 \qquad \text{for all } \mathbf{x}, \tag{2.2}$$

derivatives with respect to the radial coordinate do not appear when the operator $\mathbf{b}\cdot\nabla$ is expressed in polar coordinates. By a change of dependent variable, it is always possible, at least in a star-shaped neighborhood of **0**, to transform (2.1) so that (2.2) holds: Let

$$\psi(\mathbf{x}) = e^{(-1/2)G(\mathbf{x})}\,\phi(\mathbf{x}), \qquad G(\mathbf{x}) \equiv \int_0^1 \mathbf{x}\cdot\mathbf{b}(\lambda\mathbf{x})\,d\lambda. \tag{2.3}$$

Then

$$-\Delta\psi + \tilde{\mathbf{b}}\cdot\nabla\psi + \tilde{V}\psi = 0, \tag{2.4}$$

where

$$\tilde{\mathbf{b}} \equiv \mathbf{b} - \nabla G, \qquad \mathbf{x}\cdot\tilde{\mathbf{b}}(\mathbf{x}) = 0, \tag{2.5}$$

$$\tilde{V} \equiv V + \tfrac{1}{2}\mathbf{b}\cdot\nabla G - \tfrac{1}{2}\Delta G - \tfrac{1}{4}\nabla G\cdot\nabla G. \tag{2.6}$$

Henceforth (2.2) will be assumed.

Lemma 2.1. *Let* $\mathbf{b}(\mathbf{x})$ *be a vector field satisfying* $\mathbf{x}\cdot\mathbf{b} = 0$ *throughout a star-shaped domain about* **0**, *and define*

$$\mathbf{C}(\mathbf{x}) = \int_0^1 \mathbf{b}(\lambda\mathbf{x})\,d\lambda. \tag{2.7}$$

Then

$$\mathbf{b} \cdot \nabla\phi = \sum_{j>k=1}^{n} \left(\frac{\partial C_k}{\partial x_j} - \frac{\partial C_j}{\partial x_k}\right)\left(x_j \frac{\partial \phi}{\partial x_k} - x_k \frac{\partial \phi}{\partial x_j}\right). \tag{2.8}$$

Outline of proof. The calculation for general n generalizes certain cross-product manipulations for $n = 3$. In the language of differential forms [3], the expression on the right of (2.8) is $(\mathbf{x} \wedge \nabla\phi) \cdot d\mathbf{C}$, where the inner product is that in the space of 2-forms, and vector fields are identified with 1-forms by working in a fixed orthogonal Cartesian coordinate system in $\mathbb{R}^n$. For any $\mathbf{b}$, one can verify that

$$\mathbf{b}(\mathbf{x}) = \int_0^1 \{\mathbf{b}(\lambda\mathbf{x}) + \mathbf{x} \cdot \nabla_x[\mathbf{b}(\lambda\mathbf{x})]\}\, d\lambda. \tag{2.9}$$

If $\mathbf{x} \cdot \mathbf{b} = 0$, then applying the identity

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{B} \cdot \nabla\mathbf{A} + \mathbf{A} \cdot \nabla\mathbf{B} + (-1)^{n-1}[*(\mathbf{B} \wedge *d\mathbf{A}) + *(\mathbf{A} \wedge *d\mathbf{B})] \tag{2.10}$$

to $\mathbf{A}(\mathbf{x}) = \mathbf{x}$, $\mathbf{B}(\mathbf{x}) = \mathbf{b}(\lambda\mathbf{x})$, one deduces that the integrand in (2.9) is $*\{[*d_x\mathbf{b}(\lambda x)] \wedge x\}$. Pulling everything independent of λ outside the integral, one obtains

$$\mathbf{b}(\mathbf{x}) = * \left\{\left[*d \int_0^1 \mathbf{b}(\lambda\mathbf{x})\, d\lambda\right] \wedge \mathbf{x}\right\}. \tag{2.11}$$

The lemma follows, upon rearrangement using the properties [3, Section 2.7] of the Hodge star operator.

Equation (2.1) under study is thus reduced to the form

$$-\Delta\phi = \Gamma\phi, \qquad \Gamma \equiv -\sum_{j>k=1}^{n} \beta_{jk}(\mathbf{x})\, D_{jk} - V(\mathbf{x}), \tag{2.12}$$

where

$$D_{jk}\phi \equiv x_k \frac{\partial \phi}{\partial x_j} - x_j \frac{\partial \phi}{\partial x_k}. \tag{2.13}$$

Remark. In what follows (Definition 3.2 etc.) it is necessary to place smoothness assumptions on the potentials β_{jk} and V. These correspond to comparable conditions on the original b_j, since the constructions (2.5) and (2.8) each destroy at most one order of differentiability. In contrast, if the more elementary decomposition

$$\mathbf{b} \cdot \nabla\phi = r^{-2}(\mathbf{x} \wedge \mathbf{b}) \cdot (\mathbf{x} \wedge \nabla\phi) \equiv \sum \tilde{\beta}_{jk}(\mathbf{x})\, D_{jk}$$

were used in place of (2.8), the $\tilde{\beta}_{jk}$ for a smooth $\mathbf{b}$ would (for $n > 2$) have unacceptable singularities at the origin.

3. Function Spaces and Inverse Operators

For $\mathbf{x} \in \mathbb{R}^n$, define $r \equiv |\mathbf{x}|$ and $\boldsymbol{\xi}$ by $0 \neq \mathbf{x} = r\boldsymbol{\xi}$. Let Ω_R be the open ball $|\mathbf{x}| < R$; $L^2_{\text{loc}}(\Omega_R)$ is the space of functions square-integrable on compact subsets of Ω_R. Denote by $\|\cdot\|_S$ and $(\cdot, \cdot)_S$ the norm and inner product in $L^2(S^{n-1})$, and by $\{Y_{l,m}(\boldsymbol{\xi})\}$ an arbitrary orthonormal basis for the $N(l, n)$-dimensional subspace of spherical harmonics of order l [14; 4, pp. 126–142; 9, Chap. 6]. The term "spherical harmonic" will refer to any element of this space, not just to members of a distinguished basis.

Every function $f(\mathbf{x}) \in L^2_{\text{loc}}(\Omega_R)$ can be expanded in a series of such harmonics,

$$f(\mathbf{x}) = \sum_{l=0}^{\infty} \sum_{m=1}^{N(l,n)} \hat{f}_{l,m}(r)\, Y_{l,m}(\boldsymbol{\xi}), \tag{3.1}$$

for almost every $r < R$. Define

$$f_{l,m}(r) \equiv r^{-l} \hat{f}_{l,m}(r) \qquad (r \neq 0). \tag{3.2}$$

Similarly, let

$$\mathbf{Q}_l f(\mathbf{x}) = \sum_{m=1}^{N(l,n)} f_{l,m}(r)\, Y_{l,m}(\boldsymbol{\xi}) \equiv r^{-l}\, \mathbf{P}_l f(\mathbf{x}), \tag{3.3}$$

where $\mathbf{P}_l f$ is (for fixed r) the projection of f onto the space of spherical harmonics of order l. (The $\mathbf{x}$ dependence of $\mathbf{Q}_l f$ and the r dependence of $\|\mathbf{Q}_l f\|_S$ will usually be suppressed in the notation.)

Two classes of function spaces will be needed; the spaces in both classes consist of functions with considerable smoothness in the angular directions, but with little smoothness necessary in the radial direction. Those in the first class to be defined will serve as the underlying spaces in which expansions in the yet-to-be-constructed basis will converge; those in the other class will consist of multipliers which preserve the underlying spaces and thus can be used as potential functions.

Definition 3.1. For every real number q, A^q_R is the set of all $f \in L^2_{\text{loc}}(\Omega_R)$ for which

$$\|f\|_{q,R} \equiv \sup_{l \geq 0} \operatorname*{ess\,sup}_{0<r<R} (l+1)^q R^l \|\mathbf{Q}_l f\|_S \tag{3.4}$$

is finite.

Remark 3.1. A_R^q is a Banach space with the norm given by (3.4). For $f \in A_R^q$, the functions defined by (3.2) and (3.3) satisfy

$$\operatorname*{ess\,sup}_{0<r<R} |f_{l,m}(r)| \leqslant \operatorname*{ess\,sup}_{0<r<R} \|\mathbf{Q}_l f\|_S \leqslant \frac{\|f\|_{q,R}}{(l+1)^q R^l}. \tag{3.5}$$

In addition, the following inclusions hold: (a) If $q' > q$, then $A_R^{q'} \subset A_R^q$. (b) If $R' < R$, then functions in A_R^q restricted to $\Omega_{R'}$ belong to $A_{R'}^{q'}$ for every q'. (c) If $q > 0$, then $A_R^q \subset L^2(\Omega_R)$. The last inclusion follows from (3.5), for

$$\|f\|_S^2 = \sum_{l=0}^{\infty} r^{2l} \|\mathbf{Q}_l f\|_S^2 \leqslant \|f\|_{q,R}^2 \sum_l \left(\frac{r}{R}\right)^{2l} (l+1)^{-2q};$$

hence,

$$\|f\|_2^2 = \int_0^R r^{n-1} \|f\|_S^2 \, dr \leqslant R^n \|f\|_{q,R}^2 \sum_l (l+1)^{-2q} (2l+n)^{-1} < \infty.$$

As will be seen, the following function spaces consist of multipliers which preserve the spaces A_R^q:

DEFINITION 3.2. For every real number q, M_R^q is the set of all $V \in L^2_{\text{loc}}(\Omega_R)$ for which

$$|V|_{q,R} \equiv \operatorname*{ess\,sup}_{0<r<R} \sum_{l=0}^{\infty} 2R^l (l+1)^{|q|+1} \left[\frac{N(l,n)}{\omega_n}\right]^{1/2} \|\mathbf{Q}_l V\|_S \tag{3.6}$$

is finite. (Here, ω_n is the volume of S^{n-1}.)

Remark 3.2. M_R^q is a Banach space. If functions in M_R^q (or A_R^q) are restricted to $\Omega_{R'}$, $R' < R$, they belong to $M_{R'}^{q'}$ (and hence $A_{R'}^{q'}$) for every q'. Also, $M_R^q = M_R^{-q}$, and $M_R^q \subset M_R^{q'}$ if $|q| \geqslant |q'|$.

Remark 3.3. If $V \in M_R^q$, then $V \in L^\infty(\Omega_R)$: For a spherical harmonic of order l and unit L^2-norm, one has

$$|Y_l(\boldsymbol{\xi})| \leqslant \left[\frac{N(l,n)}{\omega_n}\right]^{1/2} \tag{3.7}$$

[14, Lemma 8, p. 14]; hence, by (3.6),

$$|V(\mathbf{x})| \leqslant \sum_{l=0}^{\infty} r^l |\mathbf{Q}_l V| \leqslant \sum_{l=0}^{\infty} R^l \left[\frac{N(l,n)}{\omega_n}\right]^{1/2} \|\mathbf{Q}_l V\|_S \leqslant |V|_{q,R}. \tag{3.8}$$

Functions in M_R^q will serve as potentials in Eq. (2.12). The main result concerning the spaces M_R^q is

THEOREM 3.1. *Let $V \in M_R^q$. For every $f \in A_R^q$, $V(\mathbf{x})f(\mathbf{x})$ is in A_R^q and*

$$\|Vf\|_{q,R} \leqslant |V|_{q,R} \|f\|_{q,R}. \tag{3.9}$$

The following lemma, useful in its own right, is needed for the proof of this theorem. This lemma generalizes facts well known for $n = 3$.

LEMMA 3.1. *Let $Y_{l_1}(\xi)$, $Y_{l_2}(\xi)$, and $Y_{l_3}(\xi)$ be spherical harmonics of orders l_1, l_2, l_3, respectively; all have norm 1 in $L^2(S^{n-1})$. Set*

$$J = \int_{S^{n-1}} Y_{l_1}(\xi)\, Y_{l_2}(\xi)\, Y_{l_3}(\xi)\, d\Omega(\xi). \tag{3.10}$$

Then

$$J = 0 \tag{3.11}$$

unless l_1, l_2, and l_3, regarded as lengths, can be arranged to form the sides of a (possibly degenerate) triangle. In general,

$$|J| \leqslant \min_{1 \leqslant j \leqslant 3} \left[\frac{N(l_j, n)}{\omega_n}\right]^{1/2}. \tag{3.12}$$

Proof. Without loss of generality, take $l_1 \leqslant l_2 \leqslant l_3$. The condition that these not form a triangle is then $l_3 > l_1 + l_2$. To see that this implies (3.11), observe that every polynomial in n variables, homogeneous of degree $l_1 + l_2$, becomes a sum of spherical harmonics of degree $l_1 + l_2$ or less when restricted to the unit sphere [18, p. 447]. Applied to a product of two harmonic polynomials, this yields

$$Y_{l_1}(\xi)\, Y_{l_2}(\xi) = \sum_{l=0}^{l_1+l_2} \sum_{m=1}^{N(l,n)} c_{l,m}\, Y_{l,m}(\xi). \tag{3.13}$$

Since $\overline{Y_{l_3}}$ is still a spherical harmonic of order l_3, (3.11) follows from (3.10), (3.13), and the orthogonality of harmonics of unequal order. If, on the other hand, $l_3 \leqslant l_1 + l_2$, (3.10) leads to

$$|J| \leqslant \sup_{\xi} |Y_{l_1}(\xi)| \int_{S^{n-1}} |Y_{l_2}|\, |Y_{l_3}|\, d\Omega(\xi),$$

so (3.12) follows by Schwarz's inequality, the normalization of the Y_l's, (3.7), and the fact that $N(l, n)$ increases with l.

Proof of Theorem 3.1. For almost every $r < R$,

$$Vf = \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} r^{l_1+l_2} (\mathbf{Q}_{l_1} V)(\mathbf{Q}_{l_2} f). \tag{3.14}$$

For r fixed, $\mathbf{Q}_{l_1}V$ and $\mathbf{Q}_{l_2}f$ are spherical harmonics of orders l_1 and l_2, respectively. If $\mathbf{P}_l$ is the projection onto the space of spherical harmonics of order l, then Lemma 3.1 implies that $\mathbf{P}_l[(\mathbf{Q}_{l_1}V)(\mathbf{Q}_{l_2}f)]=0$ unless l, l_1, and l_2 form the sides of a triangle. Thus applying $\mathbf{P}_l$ to both sides of (3.14) yields

$$\mathbf{P}_l(Vf)=\sum_{l_1=0}^{\infty}\sum_{l_2=|l-l_1|}^{l+l_1} r^{l_1+l_2}\,\mathbf{P}_l[(\mathbf{Q}_{l_1}V)(\mathbf{Q}_{l_2}f)]. \tag{3.15}$$

Furthermore, if $h\in L^2(S^{n-1})$, then $\|\mathbf{P}_l h\|_S=\sup_{Y_l}|(Y_l,h)|$, where Y_l runs over all normalized spherical harmonics of order l: thus Lemma 3.1 also implies that

$$\|\mathbf{P}_l[(\mathbf{Q}_{l_1}V)(\mathbf{Q}_{l_2}f)]\|_S\leqslant\left[\frac{N(l_1,n)}{\omega_n}\right]^{1/2}\|\mathbf{Q}_{l_1}V\|_S\|\mathbf{Q}_{l_2}f\|_S. \tag{3.16}$$

Multiply (3.15) by r^{-l}, use (3.3), take the norm of both sides, apply the triangle inequality, and use (3.16) to get

$$\|\mathbf{Q}_l(Vf)\|_S\leqslant\sum_{l_1=0}^{\infty}\sum_{l_2=|l-l_1|}^{l+l_1} r^{l_1+l_2-l}\left[\frac{N(l_1,n)}{\omega_n}\right]^{1/2}\|\mathbf{Q}_{l_1}V\|_S\,\|\mathbf{Q}_{l_2}f\|_S. \tag{3.17}$$

Multiply by $R^l(l+1)^q$ and redistribute powers of R and l_2+1:

$$\begin{aligned}R^l(l+1)^q\,\|\mathbf{Q}_l(Vf)\|_S\leqslant{}&\sum_{l_1=0}^{\infty}\left[\frac{N(l_1,n)}{\omega_n}\right]^{1/2}\|\mathbf{Q}_{l_1}V\|_S\,R^{l_1}\\ &\times\sum_{l_2=|l-l_1|}^{l+l_1}\left(\frac{r}{R}\right)^{l_1+l_2-l}\left(\frac{l+1}{l_2+1}\right)^q(l_2+1)^q R^{l_2}\|\mathbf{Q}_{l_2}f\|_S.\end{aligned} \tag{3.18}$$

Note that: (a) $r\leqslant R$; (b) $l\leqslant l_1+l_2$ (the three form the sides of a triangle); (c) by (3.4),

$$(l_2+1)^q R^{l_2}\,\|\mathbf{Q}_{l_2}f\|_S\leqslant\|f\|_{q,R}.$$

Thus the sum over l_2 in (3.18) is bounded by

$$\|f\|_{q,R}\sum_{l_2=|l-l_1|}^{l+l_1}\left(\frac{l+1}{l_2+1}\right)^q. \tag{3.19}$$

The fraction $(l+1)/(l_2+1)$ is easily seen to satisfy

$$\frac{1}{l_1+1}\leqslant\frac{l+1}{l_2+1}\leqslant l_1+1\qquad\text{when}\quad |l-l_1|\leqslant l_2\leqslant l+l_1. \tag{3.20}$$

Thus the sum in (3.19) is bounded by

$$(l_1+1)^{|q|}(|l+l_1|-|l-l_1|+1),$$

which in turn is less than $2(l_1+1)^{|q|+1}$. Substituting back into (3.18), taking the essential supremum over r and l, and using (3.6), one arrives at (3.9).

Remark 3.4. If one does not take the ess sup over r at step (c) and at the end, one obtains an r-dependent version of (3.9) which can be useful (see Corollary 3.2).

To deal with the partial differential equation (2.12) when first-order terms are present, it is necessary to investigate the action of the operator Γ, defined in (2.12), on A_R^q. The D_{jk} (see (2.13)) are generators [18, p. 452] of the n-dimensional rotation group and act only on angular variables. They map the space of spherical harmonics of order l into itself; on that space, they can be represented by anti-Hermitian matrices whose eigenvalues have magnitude l or less (since an eigenvector can be identified with a certain homogeneous polynomial of degree l). Thus $\|D_{jk}Y_l\|_S \leqslant l\,\|Y_l\|_S$ for any spherical harmonic Y_l; in addition, it is clear that for $f\in A_R^q$, $\mathbf{Q}_l(D_{jk}f)=D_{jk}(\mathbf{Q}_l f)$, so

$$\|\mathbf{Q}_l(D_{jk}f)\|_S \leqslant l\,\|\mathbf{Q}_l f\|_S. \tag{3.21}$$

COROLLARY 3.1. *Let $\beta_{jk}\in M_R^{q-1}$ and $V\in M_R^q$. The map Γ defined by (2.12) is then a bounded linear map from A_R^q to A_R^{q-1}.*

Proof. From (3.21) it is easy to see that $D_{jk}f$ belongs to A_R^{q-1}, and that D_{jk} has norm 1 as an operator from A_R^q to A_R^{q-1}. The proposition then follows from Theorem 3.1 and the inclusion $A_R^q\subset A_R^{q-1}$ (Remark 3.1).

By means of (3.21) and Remark 3.4, it is possible to obtain a useful estimate on $\|\mathbf{Q}_l\Gamma f\|_S$:

COROLLARY 3.2. *Let $\beta_{jk}\in M_R^{q-1}$ and $V\in M_R^q$. For almost every r, $0\leqslant r\leqslant R$, and every $f\in A_R^q$,*

$$\begin{aligned} &R^l(l+1)^{q-1}\,\|\mathbf{Q}_l\Gamma f\|_S \\ &\quad\leqslant \left(\sum_{j>k=1}^{n}|\beta_{jk}|_{q-1,R}+\frac{1}{l+1}|V|_{q,R}\right)\sup_{l'}\,[R^{l'}(l'+1)^q\,\|\mathbf{Q}_{l'}f\|_S]. \end{aligned}\tag{3.22}$$

Remark 3.5. Membership in M_R^q is, at root, a condition of smoothness with respect to the angular coordinates on S^{n-1}. There is virtually no restriction on the behavior of $V(\mathbf{x})$ in the radial direction, since (3.6) implies nothing about an individual Fourier component $V_{l,m}(r)$ except that it is measurable and essentially bounded. However, the restriction of V to the sphere $r=R'<R$, as an element of $L^2(S^{n-1})$, is an analytic vector, in the sense of Nelson [15], for all the operators D_{jk}; this can be shown from (3.6) and (3.21). What functions are in M_R^q? Any analytic function $V(\mathbf{x})$ which can be represented by a power series convergent for $r<R_0$ can be shown to

be in M_R^q for all q and every $R < R_0$. On the other hand, a function may be C^∞ in a neighborhood of the origin without belonging to any M_R^q; for example,

$$V(r, \theta) = \sum_{l=0}^{\infty} r^l e^{l^2 - l^4 r^2} \cos(l\theta).$$

In addition to the spaces A_R^q and M_R^q, two operators which invert $-\Delta\phi = f$ will be needed: the first, $\mathbf{G}$, yields a solution which satisfies null *initial data* at the origin; the second, $\tilde{\mathbf{G}}$, is the ordinary inverse for $-\Delta$ subject to null Dirichlet boundary conditions. These operators can be expressed in terms of polar coordinates with the aid of the integral kernels

$$\begin{aligned} g_l(r, r') &\equiv 0 && \text{for} \quad r \leqslant r', \\ &\equiv \frac{r'}{p}\left[\left(\frac{r'}{r}\right)^p - 1\right] && \text{for} \quad p \neq 0 \text{ and } r > r', \\ &\equiv r' \ln\left(\frac{r'}{r}\right) && \text{for} \quad p = 0\ [l = 0, n = 2] \text{ and } r > r', \end{aligned} \tag{3.23}$$

where $p \equiv 2l + n - 2$; and,

$$\tilde{g}_l(r, r') = g_l(r, r') - g_l(R, r'). \tag{3.24}$$

For every $f \in A_R^q$, define $\mathbf{G}f$ and $\tilde{\mathbf{G}}f$ by

$$\begin{aligned} \mathbf{Q}_l(\mathbf{G}f) &= \int_0^R g_l(r, r')\, \mathbf{Q}_l f(r', \xi)\, dr', \\ \mathbf{Q}_l(\tilde{\mathbf{G}}f) &= \int_0^R \tilde{g}_l(r, r')\, \mathbf{Q}_l f(r, \xi)\, dr'. \end{aligned} \tag{3.25}$$

The main properties of these operators are given in the next theorem.

THEOREM 3.2. *The operators* $\mathbf{G}$ *and* $\tilde{\mathbf{G}}$ *are bounded maps from* A_R^q *to* $A_R^{q'}$, $q' \leqslant q + 1$, *with norms*

$$\|\mathbf{G}\|_{q,q',R} \leqslant \tfrac{1}{2}R^2, \qquad \|\tilde{\mathbf{G}}\|_{q,q',R} \leqslant \tfrac{1}{2}R^2. \tag{3.26}$$

If $f \in A_R^q$, *then* $\mathbf{G}f$ *and* $\tilde{\mathbf{G}}f$ *have second-order derivatives in* $L^2_{\mathrm{loc}}(\Omega_R)$ *and satisfy*

$$-\Delta(\mathbf{G}f) = f, \qquad -\Delta(\tilde{\mathbf{G}}f) = f \tag{3.27}$$

in Ω_R. If $f(\mathbf{x})=0$ for all $r<R'<R$, then $\mathbf{G}f(\mathbf{x})=0$ for $r<R'$; for any $f\in A^q_R$,

$$\|\mathbf{Q}_l(\mathbf{G}f)\|_S = O(r^2). \tag{3.28}$$

Finally, on $A^q_R \cap L^2(\Omega_R)$, $\tilde{\mathbf{G}}$ coincides with the L^2-inverse of $-\Delta$ subject to null Dirichlet boundary conditions.

Proof. First consider boundedness. For almost every $r<R$,

$$\|\mathbf{Q}_l(\mathbf{G}f)\|_S^2 = \int_0^R dr' \int_0^R dr'' \left[g_l(r,r')\, g_l(r,r'') \times \int_{S^{n-1}} \overline{\mathbf{Q}_l f(r',\boldsymbol{\xi})}\, \mathbf{Q}_l f(r'',\boldsymbol{\xi})\, d\Omega(\boldsymbol{\xi}) \right].$$

Apply Schwarz's inequality to the integral over S^{n-1} and use (3.4) to get

$$\|\mathbf{Q}_l(\mathbf{G}f)\|_S^2 \leqslant \left[\int_0^R g_l(r,r')\, dr' \right]^2 \frac{\|f\|_{q,R}^2}{R^{2l}(l+1)^{2q}}. \tag{3.29}$$

The integral in (3.29) is bounded by $R^2/(2l+n)$. Take square roots in (3.29) and multiply by $R^l(l+1)^{q'}$, obtaining

$$R^l(l+1)^{q'}\, \|\mathbf{Q}_l(\mathbf{G}f)\|_S \leqslant \frac{R^2(l+1)^{q'-q}}{2l+n} \|f\|_{q,R}. \tag{3.30}$$

Since $q' \leqslant q+1$, the factors involving l are uniformly bounded by $1/2$. Taking the supremum over r and l gives (3.26) for $\mathbf{G}$. The proof for $\tilde{\mathbf{G}}$ is identical.

Now take ϕ to be $\mathbf{G}f$ or $\tilde{\mathbf{G}}f$. What must be shown next is that ϕ belongs to the Sobolev space $H^2_{\mathrm{loc}}(\Omega_R)$ and that $-\Delta\phi = f$. If the scaled Fourier components [see (3.2)] for ϕ and f are $\phi_{l,m}$ and $f_{l,m}$, then direct computation using (3.23), (3.24), and (3.25) reveals that

$$\left(-\frac{d^2}{dr^2} - \frac{2l+n-1}{r} \frac{d}{dr} \right) \phi_{l,m}(r) = f_{l,m}(r). \tag{3.31}$$

Given a C^∞ function g with support in Ω_R, calculate $(-\Delta g, \phi)$ by resolving g and ϕ into Fourier components, then integrating the radial integrals by parts, and finally using (3.31). Putting back the angular dependence then gives $(-\Delta g, \phi) = (g, f)$; thus ϕ is a distribution solution to (3.27). Standard theorems on elliptic regularity imply that $\phi \in H^2_{\mathrm{loc}}(\Omega_R)$ and that (3.27) holds in the strong sense.

Next, (3.28) and the statement prior to it are easily established using the explicit form of the kernels in (3.23) and the boundedness of $\mathbf{Q}_l f$ [Remark 3.1].

To obtain the last statement of the theorem, note that if $f \in A_R^q \cap L^2(\Omega_R)$, the scaled components of $\phi = \tilde{\mathbf{G}} f$ satisfy (3.31) and the conditions that $\phi_{l,m}(0)$ be finite and $\phi_{l,m}(R)$ equal 0. On the other hand, if ψ solves $-\Delta\psi = f$, $\psi|_{r=R} = 0$, then its scaled components also satisfy (3.31) and the boundary conditions given for the $\phi_{l,m}$. Uniqueness of the solution to the differential equation then implies $\psi = \phi$ [4, Chap. 2, Section D]. This completes the proof of Theorem 3.2.

Remark 3.6. For every $q' < q + 1$, $\mathbf{G}$ and $\tilde{\mathbf{G}}$ can be shown to map A_R^q compactly into $A_R^{q'}$.

Both operators $\mathbf{G}$ and $\tilde{\mathbf{G}}$ were constructed by the method of separation of variables. While $\tilde{\mathbf{G}}$ has the action of a conventional Green's function, $\mathbf{G}$ behaves as an outward-directed influence function or "radially retarded Green's function" [see (3.28) and the statement preceding it]. In the next section [Corollary 4.1, Definition 4.1, and Eqs. (4.18)–(4.20)], $\mathbf{G}$ will help set up a local correspondence between harmonic functions and solutions of (1.1). The "radially retarded" nature of $\mathbf{G}$ causes the solution of (1.1) to conform near $\mathbf{0}$ as closely to the corresponding harmonic function as Eq. (1.1) will allow. Thus the concrete significance, as data prescribed at $\mathbf{0}$, of the expansion coefficients in (4.18) and (4.20) is very similar in the two expansions.

4. Construction of the Basis

The machinery developed in Section 3 will now be used to solve the equation

$$
\begin{aligned}
&-\Delta\phi = \Gamma\phi,\\
&\Gamma = -\sum_{j>k=1}^{n} \beta_{jk}(\mathbf{x})\, D_{jk} - V(\mathbf{x}),\\
&D_{jk} = x_k \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_k}, \qquad \beta_{jk}(\mathbf{x}) \in M_R^{q-1},\ V(\mathbf{x}) \in M_R^q.
\end{aligned}
\tag{4.1}
$$

Suppose that ϕ is in A_R^q. By Corollary 3.1, $\Gamma\phi$ is in A_R^{q-1}, and thus by Theorem 3.2, $\mathbf{G}\Gamma\phi$ is in A_R^q and satisfies

$$-\Delta(\mathbf{G}\Gamma\phi) = \Gamma\phi. \tag{4.2}$$

Now suppose in addition that ϕ solves (4.1). One sees that $\psi \equiv \phi - \mathbf{G}\Gamma\phi$ is in A_R^q and solves Laplace's equation, $-\Delta\psi = 0$. Conversely, if $\psi \in A_R^q$ and satisfies $-\Delta\psi = 0$, and if a $\phi \in A_R^q$ exists such that $\psi = \phi - \mathbf{G}\Gamma\phi$, then $0 = -\Delta\psi = -\Delta\phi + \Delta\mathbf{G}\Gamma\phi = -\Delta\phi - \Gamma\phi$ by (4.2). Therefore, such a ϕ satisfies (4.1). Corollaries 4.1 and 4.2 will establish that this correspondence between solutions of Laplace's equation and solutions of the equation of interest is one-to-one and onto. The existence of the desired basis for solutions of (4.1) then follows from the corresponding property of Laplace's equation.

THEOREM 4.1. *For every $\lambda \in \mathbb{C}$, the operator*

$$\mathbf{K}(\lambda) \equiv \mathbf{G}(\Gamma + \lambda), \tag{4.3}$$

which maps A_R^q into itself, has only zero in its spectrum. In addition, $[1 - \mathbf{K}(\lambda)]^{-1}$ is an entire operator-valued function of λ.

Proof. For $p \geqslant 1$ and $\phi \in A_R^q$,

$$\mathbf{Q}_l(\mathbf{K}(\lambda)^p\phi)(r, \xi) = \int_0^r g_l(r, r')\, \mathbf{Q}_l((\Gamma + \lambda)\, \mathbf{K}(\lambda)^{p-1}\phi)(r', \xi)\, dr', \tag{4.4}$$

by definition of $\mathbf{G}$. Set

$$\kappa(\lambda) = \sum_{j<k=1}^{n} |\beta_{jk}|_{q-1,R} + |V|_{q,R} + |\lambda|. \tag{4.5}$$

From Corollary 3.2 it follows that

$$\begin{aligned} &R^l(l+1)^q \|\mathbf{Q}_l(\mathbf{K}(\lambda)^p\phi)\|_S \\ &\quad \leqslant \kappa(\lambda) \int_0^r (l+1)|g_l(r, r')| \sup_{l'} [R^{l'}(l'+1)^q \|\mathbf{Q}_{l'}(\mathbf{K}(\lambda)^{p-1}\phi)\|_S]\, dr'. \end{aligned} \tag{4.6}$$

If $2l + n - 2 > 0$, (3.23) implies $(l+1)|\, g_l(r, r')| \leqslant r'$; in the case $n = 2$, $l = 0$, (3.23) implies

$$|\, g_0(r, r')| \leqslant r' \ln\left(\frac{R}{r'}\right).$$

Thus we have the uniform bound

$$(l+1)|\, g_l(r, r')| \leqslant r' \left[1 + \ln\left(\frac{R}{r'}\right)\right] \equiv g(r'), \tag{4.7}$$

$$\int_0^R g(r')\, dr' = \frac{3}{4} R^2. \tag{4.8}$$

Together (4.6) and (4.7) give the inequality

$$\sup_l \,[R^l(l+1)^q \|\mathbf{Q}_l(\mathbf{K}(\lambda)^p\phi)\|_S]$$
$$\leqslant \kappa(\lambda) \int_0^r g(r') \sup_l \,[R^l(l+1)^q \|\mathbf{Q}_l(\mathbf{K}(\lambda)^{p-1}\phi)\|_S]\, dr' \qquad (4.9)$$

for all r, $0 \leqslant r \leqslant R$; the quantity in brackets on the right is evaluated at r'. The estimate (4.9) can be expanded recursively into a p-fold integral; the key observation is that, because of the "radially retarded" nature of $\mathbf{G}$, the upper limit of integration in each interior integral is not R, but the next outermost integration variable. Therefore, by a standard argument from the theory of Volterra integral equations,

$$\|\mathbf{K}(\lambda)^p\phi\|_{q,R} \leqslant \frac{\kappa(\lambda)^p}{p!} \left[\int_0^R g(r')\, dr'\right]^p \|\phi\|_{q,R}, \qquad (4.10)$$

and hence [by (4.8)]

$$\|\mathbf{K}(\lambda)^p\|_{q,q,R} \leqslant \frac{1}{p!} [R^2\kappa(\lambda)]^p \qquad (p \geqslant 0). \qquad (4.11)$$

An immediate consequence of (4.11) is that the spectral radius of $\mathbf{K}(\lambda)$ is 0, so that 0 is the only point in its spectrum. A second consequence is that the Neumann series,

$$[1 - \mathbf{K}(\lambda)]^{-1} = \sum_{p=0}^{\infty} \mathbf{K}(\lambda)^p, \qquad (4.12)$$

is uniformly convergent in λ on every compact subset of $\mathbb{C}$. Since each truncation of this series is a polynomial in λ, it follows [8, Chap. 3] that $[1 - \mathbf{K}(\lambda)]^{-1}$ is analytic in the interior of every compact set and therefore is an entire function.

Corollary 4.1. *If ϕ is assumed to lie in A_R^q, then*

$$\psi = \phi - \mathbf{G}\Gamma\phi \qquad (4.13)$$

is also in A_R^q. Conversely, if ψ is assumed to lie in A_R^q, then

$$\phi = (1 - \mathbf{G}\Gamma)^{-1}\psi \qquad (4.14)$$

is a well-defined member of A_R^q. In either case, ϕ is a solution of (4.1) *if and only if $-\Delta\psi = 0$.*

Proof. The inverse operator required in (4.14) exists, by Theorem 4.1

with $\lambda = 0$. The rest of the argument was given at the beginning of this section.

The significance of this result hinges on the question of whether *all* solutions of the two partial differential equations belong to A_R^q spaces. The resolution of this question in the case of (4.1) is the subject of later theorems in this section. For Laplace's equation the answer is well known. The harmonic polynomials

$$H_{l,m}(\mathbf{x}) \equiv r^l Y_{l,m}(\boldsymbol{\xi}) \tag{4.15}$$

belong to A_R^q for *every* q and R, and satisfy $-\Delta H_{l,m} = 0$ [14, pp. 1–5; 9, Sec. 6.1]. Any harmonic function can be written as a series in the $H_{l,m}$; in other words, the $f_{l,m}(r)$ in (3.2) are constants, $a_{l,m}$, if (and only if) $\Delta f = 0$. The classical radius of convergence of the series,

$$\rho = (\limsup |a_{l,m}|^{1/l})^{-1}, \tag{4.16}$$

marks the boundary of the set of values of R for which $f \in A_R^q$ (for any q). A sequence $\{a_{l,m}\}$ for which $\rho = 0$ does not correspond to any solution of $\Delta f = 0$ (even in a distributional sense, in view of the regularity theorem for Laplace's equation).

DEFINITION 4.1. For each n-dimensional spherical harmonic $Y_{l,m}$, $\Phi_{l,m}$ is the solution of (4.1) corresponding, via Corollary 4.1, to the harmonic polynomial (4.15):

$$\Phi_{l,m} \equiv (1 - \mathbf{G}\Gamma)^{-1} H_{l,m}. \tag{4.17}$$

The set $\{\Phi_{l,m}\}$ is the desired basis for all solutions of (4.1) in neighborhoods of the origin. (The remaining theorems of this section justify this claim.)

Remark 4.1. If $\Gamma = \Gamma' + \lambda$ ($\lambda \in \mathbb{C}$, Γ' independent of λ), then the $\Phi_{l,m}$ are entire functions of λ, by virtue of Theorem 4.1. This observation is expected to be of use in the eigenfunction expansion theory, in analogy with [10].

The foregoing remarks about expansion of harmonic functions, combined with (4.17) and (4.14), provide a representation of a solution of (4.1) [in A_R^q] as a series in the $\Phi_{l,m}$:

THEOREM 4.2. *Every $\phi \in A_R^q$ which satisfies* (4.1) *can be represented by a series expansion,*

$$\phi = \sum_{l=0}^{\infty} \sum_{m=1}^{N(l,n)} a_{l,m} \Phi_{l,m}. \tag{4.18}$$

The $a_{l,m}$ are constants, given by the formula

$$a_{l,m} = r^{-l} \int_{S^{n-1}} (\phi - \mathbf{G}\Gamma\phi)(r, \xi)\, \overline{Y_{l,m}(\xi)}\, d\Omega(\xi), \tag{4.19}$$

where r can be any number in $0 < r < R$. Let $\varepsilon > 0$. If $\beta_{jk} \in M_R^{|q-1|+\epsilon}$, $V \in M_R^{|q|+\epsilon}$, then the series converges to ϕ in $A_R^{q-\epsilon}$, and converges in $L^\infty(\Omega')$ for every compact subset $\Omega' \subset \Omega_R$.

Proof. By Corollary 4.1, $\psi \equiv \phi - \mathbf{G}\Gamma\phi$ belongs to A_R^q and is harmonic in Ω_R. Its expansion in spherical harmonics is

$$\psi = \sum_{l=0}^{\infty} r^l \left[\sum_{m=1}^{N(l,n)} a_{l,m} Y_{l,m}(\xi) \right], \tag{4.20}$$

with the $a_{l,m}$ given by (4.19). Consider the partial sums

$$\psi_L = \sum_{l=0}^{L} \left[\sum_{m=1}^{N(l,n)} a_{l,m} H_{l,m}(\xi) \right],$$

$$\phi_L = \sum_{l=0}^{L} \left[\sum_{m=1}^{N(l,n)} a_{l,m} \Phi_{l,m}(\xi) \right].$$

From (4.17) and linearity,

$$\phi_L = (1 - \mathbf{G}\Gamma)^{-1} \psi_L.$$

It is clear from the definition of the norms involved that

$$\| \psi - \psi_L \|_{q-\epsilon,R} \leqslant (L+1)^{-\epsilon} \| \psi \|_{q,R}.$$

By Theorem 4.1 and the hypotheses on β_{jk} and V, $(1 - \mathbf{G}\Gamma)^{-1}$ is a bounded map from $A_R^{q-\epsilon}$ to $A_R^{q-\epsilon}$, so

$$\| \phi - \phi_L \|_{q-\epsilon,R} \leqslant C(R, q, \varepsilon)(L+1)^{-\epsilon} \| \psi \|_{q,R}. \tag{4.21}$$

Hence, $\phi_L \to \phi$ in $A_R^{q-\epsilon}$ if $\varepsilon > 0$.

To prove convergence in $L^\infty(\Omega')$, it is sufficient to prove such convergence on every $\Omega_{R'}$, $R' < R$. Since $\mathbf{Q}_l(\phi - \phi_L)$ is a spherical harmonic of order l, inequality (3.7) implies

$$|\mathbf{Q}_l(\phi - \phi_L)(r, \xi)| \leqslant \left[\frac{N(l,n)}{\omega_n} \right]^{1/2} \| \mathbf{Q}_l(\phi - \phi_L) \|_S. \tag{4.22}$$

The definition of $\|\cdot\|_{q-\epsilon,R}$ and the inequalities (4.21), (4.22) give

$$r^l|\mathbf{Q}_l(\phi-\phi_L)(r,\xi)| \leqslant \left[\frac{N(l,n)}{\omega_n}\right]^{1/2} \frac{C(R,q,\varepsilon)\|\psi\|_{q,R}}{(L+1)^{\epsilon}(l+1)^{q-\epsilon}}\left(\frac{R'}{R}\right)^l \tag{4.23}$$

for $r \leqslant R'$. Since the right side of (4.23) grows no worse than a fixed power of $l+1$ multiplied by the exponentially decreasing factor $(R'/R)^l$, the expansion of $\phi-\phi_L$ in spherical harmonics is uniformly convergent. Indeed, this expansion is

$$\phi-\phi_L = \sum_{l=0}^{\infty} r^l \mathbf{Q}_l(\phi-\phi_L),$$

so (4.23) implies

$$|\phi-\phi_L| \leqslant (L+1)^{-\epsilon} C(R,q,\varepsilon)\, \|\psi\|_{q,R} F(q,R'/R),$$

where $F(q,R'/R)$ is the sum over l of the various factors remaining on the right of (4.23). It follows immediately that

$$\lim_{L\to\infty} \|\phi-\phi_L\|_\infty = 0,$$

where the L^∞ norm is over $\Omega_{R'}$. This completes the proof of Theorem 4.2.

Remark 4.2. If $\beta_{jk}=0$ and $q>0$, then the theorem holds with $0<\varepsilon\leqslant q$, $V\in M_R^q$.

What remains is to show that all solutions of (4.1) satisfying conventional regularity conditions can be represented by series of the form (4.18). Attention will be confined to those solutions of (4.1) which belong to the Sobolev space $H^2(\Omega_R)$; the reasons for this are: (a) For sufficiently smooth boundary data, the solution to the Dirichlet problem associated with (4.1) will be in $H^2(\Omega_R)$. (b) Every function which solves (4.1) in a region $\Omega\supset\bar{\Omega}_R$ will be in $H^2(\Omega_R)$. (See [5, Chap. 8].) In the remainder of this section, it will be shown that for all sufficiently small R, solutions of (4.1) belonging to $H^2(\Omega_R)$ also belong to $A_R^{3/2}$. Hence, Theorem 4.2 applies to such solutions (Corollary 4.3).

Let $\phi\in H^2(\Omega_R)$ and suppose that ϕ solves (4.1) in Ω_R. Because ϕ belongs to $H^2(\Omega_R)$, its restriction to $r=R$ is well defined in the usual Sobolev sense; that is, it has a well-defined "trace" which belongs to $H^{3/2}(S^{n-1})$ [12, Theorem 9.4, p. 41]. This, in turn, implies that the Dirichlet problem,

$$\Delta\psi = 0,$$
$$\psi|_{r=R} = \phi|_{r=R}, \tag{4.24}$$

has a solution ψ which belongs to $H^2(\Omega_R)$. (See [12, pp. 188–189].) More, however, can be said:

LEMMA 4.1. *If* ψ *solves the Dirichlet problem* (4.24), *then* $\psi \in A_R^{3/2} \cap H^2(\Omega_R)$.

Proof. Expand $\phi(R, \xi) = \phi|_{r=R}$ in spherical harmonics:

$$\phi(R, \xi) = \sum_{l=0}^{\infty} R^l \left(\sum_{m=1}^{N(l,n)} f_{l,m} Y_{l,m}(\xi) \right), \tag{4.25}$$

where the $f_{l,m}$ are constants. From (4.25) it is obvious that the solution to (4.24) is

$$\psi(r, \xi) = \sum_{l=0}^{\infty} r^l \left(\sum_{m=1}^{N(l,n)} f_{l,m} Y_{l,m}(\xi) \right), \tag{4.26}$$

and hence that

$$\|\mathbf{Q}_l \psi\|_S^2 = \sum_{m=1}^{N(l,n)} |f_{l,m}|^2. \tag{4.27}$$

As was noted earlier, $\phi(R, \xi)$ is in $H^{3/2}(S^{n-1}) = \mathrm{Dom}((-\Delta_s)^{3/4})$ (see [12, p. 37]), where Δ_s is the Laplace–Beltrami operator on S^{n-1}; hence $\|(-\Delta_s)^{3/4} \phi(R, \xi)\|_S$ is finite. But (4.27) and the equation $-\Delta_s Y_{l,m} = l(l+n-2)\, Y_{l,m}$ imply

$$\begin{aligned} &\|(-\Delta_s)^{3/4} \phi(R, \xi)\|_S^2 + \|\phi(R, \xi)\|_S^2 \\ &\quad = \sum_{l=0}^{\infty} R^{2l} \{[l(l+n-2)]^{3/2} + 1\} \|\mathbf{Q}_l \psi(r, \xi)\|_S^2. \end{aligned} \tag{4.28}$$

The definition of the norm in A_R^q and a straightforward estimate obtained from (4.28) yield (for $n \geqslant 2$)

$$\|\psi\|_{3/2,R} \leqslant 2(\text{left side of (4.28)})^{1/2},$$

thus $\psi \in A_R^{3/2}$. That $\psi \in H^2(\Omega_R)$ has already been noted, so the proof is complete.

Using the harmonic function ψ, (4.1) may be rewritten as

$$-\Delta(\phi - \psi) = \Gamma(\phi - \psi) + \Gamma\psi. \tag{4.29}$$

The difference $\phi - \psi$ is in $H^2(\Omega_R)$ and satisfies $(\phi - \psi)|_{r=R} = 0$; the function $\Gamma\psi$ belongs to $A_R^{1/2} \subseteq L^2(\Omega_R)$. (See Remark 3.1(c).) Thus $\tilde{\mathbf{G}}$, which is the Green's function for the Dirichlet problem defined by (3.24) and (3.25), may be applied to both sides of (4.29). Doing this and rearranging terms in the resulting equation yields

$$\phi - \psi - \tilde{\mathbf{G}}\Gamma(\phi - \psi) = \psi', \tag{4.30}$$

where $\psi' = \tilde{\mathbf{G}}\Gamma\psi$. The function ψ' belongs to $H^2(\Omega_R)$ because $\Gamma\psi \in L^2(\Omega_R)$ and, as is well known, $\tilde{\mathbf{G}}$ maps $L^2(\Omega_R)$ continuously into $H^2(\Omega_R)$. In addition, Theorem 3.2 and the fact that $\Gamma\psi$ is in $A_R^{1/2}$ imply that $\psi' \in A_R^{3/2} \cap H^2(\Omega_R)$. Thus Eq. (4.30) may be regarded as an inhomogeneous equation for $\phi - \psi$ in either $A_R^{3/2}$ or $H^2(\Omega_R)$. Taking this dual view of (4.30) leads to the following result:

THEOREM 4.3. *If* $1 - \tilde{\mathbf{G}}\Gamma$ *has a bounded inverse on* $A_R^{3/2}$ *and if* $\tilde{\mathbf{G}}\Gamma$ *has no eigenfunction in* $H^2(\Omega_R)$ *corresponding to the eigenvalue* 1, *then every solution to* (4.1) *which is in* $H^2(\Omega_R)$ *is also in* $A_R^{3/2}$.

Proof. Because ψ' is in $A_R^{3/2}$ and $1 - \tilde{\mathbf{G}}\Gamma$ is boundedly invertible, $\chi \equiv (1 - \tilde{\mathbf{G}}\Gamma)^{-1}\psi'$ exists, is in $A_R^{3/2}$, and satisfies $\chi = \tilde{\mathbf{G}}\Gamma\chi + \psi'$. Corollary 3.1 and Remark 3.1(c) then imply $\Gamma\chi \in A_R^{1/2} \subseteq L^2(\Omega_R)$. Since $\tilde{\mathbf{G}}$ maps $L^2(\Omega_R)$ into $H^2(\Omega_R)$, it is clear that $\tilde{\mathbf{G}}\Gamma\chi$ and, hence, $\tilde{\mathbf{G}}\Gamma\chi + \psi' = \chi$ are in $H^2(\Omega_R)$. Since $\tilde{\mathbf{G}}\Gamma$ has no eigenfunction in $H^2(\Omega_R)$ corresponding to the eigenvalue 1, χ is the only solution to $\chi - \tilde{\mathbf{G}}\Gamma\chi = \psi'$ in $H^2(\Omega_R)$. On the other hand, $\phi - \psi$ satisfies (4.30) and is in $H^2(\Omega_R)$, so $\phi - \psi = \chi$. From this and the fact that both ψ and χ are in $A_R^{3/2}$, it follows that $\phi \in A_R^{3/2}$. This finishes the proof.

What will be done next is to show that the two conditions given in Theorem 4.3 hold for all sufficiently small R. The estimates given in the next two lemmas will be needed.

LEMMA 4.3. *If* Γ *is given by* (4.1) *with* $q = 3/2$, *then*

$$\|\tilde{\mathbf{G}}\Gamma\|_{3/2,3/2,R} \leqslant \tfrac{1}{2}R^2\sigma(R), \tag{4.31}$$

where

$$\sigma(R) = \sum_{j>k=1}^{n} |\beta_{jk}|_{1/2,R} + |V|_{3/2,R}.$$

Proof. Apply Theorem 3.2 and Corollary 3.2.

LEMMA 4.4. *Let* Γ *and* σ *be as in Lemma* 4.3. *If* $\chi \in H^2(\Omega_R)$ *and* $\chi|_{r=R} = 0$, *then*

$$|\langle\Gamma\chi, \chi\rangle| \leqslant \frac{2R^2}{z_n}\sigma(R)\langle -\Delta\chi, \chi\rangle, \tag{4.32}$$

where z_n *is the smallest positive zero of the Bessel function* $J_{(1/2)(n-2)}(z)$.

Proof. In $|\langle \Gamma\chi, \chi\rangle|$, replace Γ by the expression given in (4.1). Apply Schwarz's inequality and the triangle inequality to the result, then use the inequality $\|fg\|_2 \leqslant \|f\|_\infty \|g\|_2 \leqslant |f|_{q,R}\, \|g\|_2$, which holds for all $f \in M_R^q$ and $g \in L^2(\Omega_R)$ (see Remark 3.3), to get

$$|\langle \Gamma\chi, \chi\rangle| \leqslant \|\chi\|_2 \left[\sum_{j>k=1}^{n} |\beta_{jk}|_{1/2,R}\, \|D_{jk}\chi\|_2 + |V|_{3/2,R}\, \|\chi\|_2 \right]. \tag{4.33}$$

From the definition [in (4.1)] of D_{jk}, one has $\|D_{jk}\chi\|_2 \leqslant 2R\, \|\nabla\chi\|_2$. In addition, integrating $\langle -\Delta\chi, \chi\rangle$ by parts gives $\langle -\Delta\chi, \chi\rangle = \|\nabla\chi\|_2^2$; minimizing $\langle -\Delta\chi, \chi\rangle$, subject to the constraints that $\chi \in H^2(\Omega_R)$, $\chi|_{r=R} = 0$, and $\|\chi\|_2$ be constant, yields the inequality $\|\chi\|_2^2 \leqslant (R/z_n)^2 \langle -\Delta\chi, \chi\rangle$ (since $(z_n/R)^2$ is the smallest eigenvalue of $-\Delta$ in Ω_R with the null Dirichlet boundary condition). Moreover, because z_n increases with n and $z_2 > 2$, it follows that $1/2z_n < 1$. Finally, combine the equation and inequalities discussed above with (4.33); the result is (4.32). This ends the proof.

The estimates in the two lemmas provide:

COROLLARY 4.2. *Fix $R_0 > 0$ and suppose V and β_{jk} belong to $M_{R_0}^{3/2}$ and $M_{R_0}^{1/2}$, respectively. There exists $R_1 \leqslant R_0$ such that for all $R < R_1$, every $\phi \in H^2(\Omega_R)$ which solves* (4.1) *belongs to $A_R^{3/2}$.*

Proof. From the form of $\sigma(R)$ given in Lemma 4.3, it is clear that $\sigma(R)$ is an increasing function of R; thus $\sigma(R) \leqslant \sigma(R_0)$ for all $R \leqslant R_0$. Put $R_1 = \sigma(R_0)^{-1/2} \cdot \min(\sqrt{2}, \sqrt{z_n/2})$, so that $\frac{1}{2}R^2\sigma(R) < 1$ and $(2/z_n)\, R^2\sigma(R) < 1$ for all $R < R_1$. Lemma 4.3 then implies that $\|\tilde{\mathbf{G}}\Gamma\|_{3/2,3/2,R} < 1$, so $1 - \tilde{\mathbf{G}}\Gamma$ is boundedly invertible in $A_R^{3/2}$. Lemma 4.4 also implies that $|\langle \Gamma\chi, \chi\rangle| < \langle -\Delta\chi, \chi\rangle$ for all $\chi \in H^2(\Omega_R)$ which satisfy $\chi|_{r=R} = 0$. On the other hand, if χ were an eigenfunction of $\tilde{\mathbf{G}}\Gamma$ with eigenvalue one, it is easy to see that $\langle \Gamma\chi, \chi\rangle = \langle -\Delta\chi, \chi\rangle$, as $-\Delta\chi = \Gamma\chi$. Thus, $\tilde{\mathbf{G}}\Gamma$ can have no such eigenfunctions. The desired conclusions follow from Theorem 4.3.

Convergence of the series associated with ϕ can now be proved.

COROLLARY 4.3. *With the assumptions and notation of Corollary* 4.2, *for each $R < R_1$ and every solution ϕ to* (4.1) *in $H^2(\Omega_R)$, the series*

$$\sum_{l=0}^{\infty} \sum_{m=1}^{N(l,n)} a_{l,m}\, \Phi_{l,m} \tag{4.34}$$

converges to ϕ in $A^{3/2-\epsilon}$ (for all $\varepsilon > 0$) and converges uniformly to ϕ on compact subsets of Ω_R. Here $\{\Phi_{l,m}\}$ is the basis constructed in (4.17), *and the $a_{l,m}$ for ϕ are given by* (4.19).

Proof. Corollary 4.2 implies $\phi \in A_R^{3/2}$. By Remark 3.2, V and β_{jk} belong to M_R^q for every q. Apply Theorem 4.2.

Corollary 4.3 is the result promised earlier: Every solution to (4.1) which satisfies conventional regularity conditions has a convergent series expansion of the form (4.34), at least in Ω_R for R sufficiently small. The set $\{\Phi_{l,m}\}$ very clearly plays the role of a basis for the space of solutions to (4.1).

Remark 4.3. If $\beta_{jk} = 0$, then in view of Remark 3.6, $\tilde{\mathbf{G}}\Gamma$ maps $A_R^{3/2}$ compactly into $A_R^{3/2}$. Using this compactness it is possible to obtain Corollaries 4.2 and 4.3 with $R_1 = R_0$. The lengthy details of the argument are omitted.

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